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SOLVABLE MODELS IN TWO-DIMENSIONAL N=2 THEORIES

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ABSTRACT

N=2 supersymmetric field theories in two dimensions have been extensively studied in the last few years. Many of their properties can be determined along the whole renormalization group flow, like their coupling dependence and soliton spectra. We discuss here several models which can be solved completely, when the number of superfields is taken to be large, by studying their topological-antitopological fusion equations. These models are the $\mathbb{C}P^{N-1}$model, σ-models on Grassmannian manifolds, and certain perturbed $N=2$ Minimal model.

1. Introduction

Two-dimensional quantum field theories have been studied extensively in the last few years, both at the conformal point and off criticality for massive and massless flows. More recently, there has been an extensive exploration of two-dimensional field theories with $N=2$ supersymmetry. These theories play for example a fundamental role in the construction of $N=1$ string vacua and the study of string compactification. Many of the models admit Landau-Ginsburg-type actions characterized by a superpotential which obeys non-renormalization theorems, thus making them easier to study. The superpotential encodes the chiral ring of operators (the topological data of the theory) and many of the properties of the full quantum theory can be determined through it by defining a metric, the inner product on the space of supersymmetric ground-states (a generalization of Zamolodchikov's metric) which satisfies a set of differential equations on the space of couplings, the topological-antitopological (tt^*) fusion equations. This metric, and a new index related to it, are defined along the whole renormalization group flow and give information about the soliton spectrum and other characteristics of the model such as the scale and coupling dependence. The new index, a non-perturbative quantity, is calculable exactly in any two-dimensional $N=2$ theory, whether or not it is integrable. From it a sort of c function on the space of couplings can also be defined, which decreases monotonically along the flow. The 'new index' can be calculated either from the ground-state metric through the tt^* equations, or by solving the TBA equations if the S-matrix of the model is known. The tt^* fusion equations have proven difficult to solve and their equivalence with the TBA approach has only been showed numerically for some simple cases.

The tt^* formalism has a wide range of applications in two-dimensional systems. Apart from the models we will be discussing here, many of the properties of two-dimensional polymers (self-avoiding random walks) can be studied. The index turns out to be the partition function for a single polymer which loops around a cylinder and it gives the scaling function for the number of such configurations.

Here, we investigate several models where the tt^* equations can be solved completely analytically. Quantum field theories usually become tractable when the number of fields in the model is very large. For example, the large N $\mathbb{C}P^{N-1}$model has been solved: to leading order in $1/N$, S-matrix elements are given by summing tree diagrams, while bulk quantities like the free energy are calculable by simply extremizing an effective action.

We show here that the tt^* methods show comparable simplifications in this limit. The models we will be considering are the $\mathbb{C}P^{N-1}$ model, σ-models on Grassmannian target manifolds $\mathbb{G}(k,N)$ and certain perturbed A_n Minimal models.

2. The ground-state metric for N=2 theories and the new index

The situation governed by the tt^* equations is the following. We have a $d=2$, $N=2$ supersymmetric field theory quantized on a Euclidean manifold, with metric and boundary conditions preserving $N=2$ global supersymmetry. We assume there are a discrete set of supersymmetric ground states, that at least one dimension is compact, and that a Hamiltonian defined on a compact hypersurface has a gap. Given all this, certain correlation functions can be reduced to sums over the ground states.

One way to describe the tt^* results is that they answer the question: from the topological data of an $N=2$ theory, in which only the ground-states survive, what can one reconstruct about the original theory?

Let us then consider an $N=2$ theory on a cylinder, where the long dimension has length L (which will go to infinity), and the compact dimension has circumference β.

In two dimensions the $N=2$ supersymmetry algebra takes the form

$$\begin{aligned}
Q_L^{+2} = Q_L^{-2} = Q_R^{+2} &= Q_R^{-2} = & \{Q_L^+, Q_R^-\} &= \{Q_L^-, Q_R^+\} = 0 \\
\{Q_L^+, Q_R^+\} &= 2\Delta & \{Q_L^-, Q_R^-\} &= 2\Delta^* \\
\{Q_L^+, Q_L^-\} &= H - P & \{Q_R^+, Q_R^-\} &= H + P \\
[F, Q_L^\pm] &= \pm Q_L^\pm & [F, Q_R^\pm] &= \mp Q_R^\pm \\
Q_L^{+\dagger} &= Q_L^- & (Q_R^+)^\dagger &= {Q_R}^- \qquad (1) \\
Q^\pm \equiv \frac{1}{\sqrt{2}}&(Q_L^\pm + Q_R^\pm) & \{Q^-, Q^+\} &= H. \qquad (2)
\end{aligned}$$

F, the 'fermion number,' is the conserved $U(1)$ charge, and $Q_L^\pm$ and $Q_R^\pm$ are left and right supercharges. In non-compact space one can have a non-zero central term Δ, which will come in below.

The most basic elements particular to a given $N = 2$ theory are the chiral and anti-chiral rings. The chiral operators ϕ_i satisfy $[Q^+, \phi_i] = 0$, and the anti-chiral operators $\bar{\phi}_i$ satisfy $[Q^-, \bar{\phi}_i] = 0$. The chiral ring is defined in terms of the operator product algebra as

$$\phi_i \phi_j = \sum_k C_{ij}^k \phi_k + [Q^+, \Lambda]. \tag{3}$$

Since the derivative of any operator (and the stress tensor itself) is a descendant under Q^+ (and Q^-), the positions of the operators on the left hand side do not matter. The anti-chiral ring will have structure constants $\bar{C}_{ij}^k$.

Equally important are the supersymmetric (Ramond) ground states

$$H|a\rangle = Q^{\pm}|a\rangle = 0. \tag{4}$$

We could make a correspondence between these and chiral fields by choosing a canonical ground state $|0\rangle$. Then we can identify

$$\phi_i|0\rangle = |i\rangle + Q^+|\Lambda\rangle. \tag{5}$$

Finally we could project on the true ground state by applying an operator like $\lim_{T\to\infty} \exp -HT$. We could also do this with anti-chiral fields $\bar{\phi}_i$, producing states to be called $|\bar{i}\rangle$. The structure constants C_{ij}^k then also give the action of the chiral operators on the ground states:

$$\phi_i|j\rangle = C_{ij}^k|k\rangle + Q^+|\psi\rangle. \tag{6}$$

This construction is not completely satisfactory because it is not clear that the correspondence is one to one; furthermore it depended on the choice of $|0\rangle$. Both problems are dealt with by making a correspondence using spectral flow.[1] In principle, this constructs the state $|i\rangle$ by doing a path integral on a hemisphere with an insertion of ϕ_i. We need spectral flow to put this state in the Ramond sector, and we can think of it as turning on a $U(1)$ gauge field coupled to the fermion number current, with holonomy $e^{i\pi}$ on the boundary. We can then take as $|0\rangle$ the state produced by inserting the identity operator $\phi_0 \equiv 1$, and non-degeneracy of the two-point function $\langle \bar{\phi}_i \phi_j \rangle$ will imply that the correspondence is one to one.

Now we take $|i\rangle$ and $|\bar{j}\rangle$ to denote the basis of ground states corresponding to the fields ϕ_i and $\phi_{\bar{j}}$. CPT will relate $|i\rangle$ to a state $\langle \bar{i}|$ so the usual Hilbert space metric will be the hermitian

$$g_{i\bar{j}} = \langle \bar{j}|i\rangle. \tag{7}$$

Another structure present in the theory is the "real structure" M expressing one basis in terms of the other:

$$\langle \bar{i}| = \langle j|M_{\bar{i}}^j \tag{8}$$

CPT implies $MM^* = 1$.
A combination of these produces the 'topological' metric η: $\quad g_{i\bar{k}} = \eta_{ij} M_{\bar{k}}^j$. $\quad$ This is the two-point function in the topologically twisted theory.

Supersymmetry-preserving perturbations of the action are of two types. In general we need to write a commutator with all four supercharges (or integral $d^4\theta$) to preserve all supersymmetries. However we can also write

$$\delta S = \sum_i \int d^2x \quad \delta t_i \{Q_R^-, [Q_L^-, \phi_i]\} + \delta \bar{t}_i \{Q_R^+, [Q_L^+, \bar{\phi}_i]\}. \tag{9}$$

where ϕ_i and $\bar{\phi}_i$ are chiral and anti-chiral fields. A perturbation which can only be written in this form is called an F term; the others are D terms.

In the spirit of the non-abelian Berry's phase, define the gauge connection

$$A_i{}^j{}_k = g^{j\bar{j}'} \langle \bar{j}' | \partial_i | k \rangle \tag{10}$$

and its conjugate. By definition, the metric g is covariantly constant with respect to the derivatives $\quad D_i = \partial_i - A_i, \quad \bar{D}_i = \bar{\partial}_{\bar{i}} - \bar{A}_{\bar{i}}$.
Then, we might expect covariant combinations like the curvature to be especially simple. Writing these out explicitly and manipulating the supercharges gives

$$[D_i, D_j] = [\bar{D}_{\bar{i}}, \bar{D}_{\bar{j}}] = 0 \tag{11}$$

and for the mixed terms, one finds

$$[D_i, \bar{D}_{\bar{j}}] = -\beta^2 [C_i, \bar{C}_{\bar{j}}], \tag{12}$$

a differential equation for the metric. By (11) one can choose a basis in which $\bar{A}_i = 0$, so $A_i = g^{-1}\partial_i g$, and (14) becomes

$$\bar{\partial}_{\bar{j}}(g \partial_i g^{-1}) = \beta^2 [C_i, g C_{\bar{j}}^{\dagger} g^{-1}] \tag{13}$$

These are the tt^* equations, which given enough boundary conditions determine the metric g. Different models with the same chiral ring can have different metrics: thus the boundary conditions are a crucial part of the story. In the cases considered in detail,[1,4,2] the dependence on one relevant coupling is studied. Let this define a mass scale m; then small βm is weak coupling and this limit of the metric can be found using semiclassical techniques. The large βm boundary conditions are even simpler and are best explained in terms of the 'new index' (see below).

There is another observable depending only on F couplings[2]. Although it is simply related to the metric it has a clearer physical interpretation. It is modeled after the index $\mathrm{Tr}(-1)^F e^{-\beta H}$, which is completely independent of finite perturbations of the theory for $N \geq 1$ supersymmetric theories in any dimension.[10] This index has been very useful in providing criteria for supersymmetry breaking. For an $N = 2$ theory in two dimensions, the new observable is the 'new index' $\mathrm{Tr}\, F(-1)^F e^{-\beta H}$, with F the 'Fermion number.'

The new index is actually a matrix since the boundary conditions at spatial infinity can be any vacuum of the theory. Let the left vacuum be a and the right one b, and consider the matrix elements

$$Q_{ab} = \frac{i\beta}{L} \mathrm{Tr}_{\ ab}\ (-1)^F\ F\ e^{-\beta H}. \tag{14}$$

In [2] it is shown that the matrix Q is imaginary and hermitian, and that

$$Q_{ab} = i(\beta g \partial_\beta g^{-1} + n)_{ab} \tag{15}$$

where n is the coefficient of the chiral anomaly.

This quantity is particularly suited for extracting the soliton spectrum and other low temperature properties of the model. The simplest case is a model with a mass gap; clearly Q will be exponentially small in βm and typically each of the leading terms in an expansion in $\exp -\beta m$ is the contribution of a single massive particle saturating the Bogomolnyi bound $m = |\Delta|$. [5,2,8]

3. $\mathbb{C}P^{N-1}$models

We now apply the formalism to the supersymmetric $\mathbb{C}P^{N-1}\sigma$-models.
Non-linear σ-models define maps from spacetime into a riemannian target manifold M. Supersymmetric $d = 2$ σ-models exist for any target manifold. If the target manifold and metric is Kähler,[9] the model will be $N = 2$.
A manifestly $N = 2$ invariant superspace Lagrangian is

$$\mathcal{L} = \frac{1}{2}\int dx\ d^2\theta\ d^2\bar{\theta}\ \ K(\Phi, \Phi^\dagger), \tag{16}$$

where K is the Kähler potential and Φ_i are complex chiral superfields

$$\Phi_i = \Phi_i(x, \theta, \bar{\theta}) = \varphi_i(x) + \sqrt{2}\epsilon_{\alpha\beta}\theta^\alpha \Psi_i^\beta(x) + \epsilon_{\alpha\beta}\theta^\alpha\theta^\beta F_i(x)\ . \tag{17}$$

Any term in K which is globally defined on M is a D term, and conversely two choices of K for which the Kähler forms $J = dz^i \wedge d\bar{z}^{\bar{j}}\partial_i\bar{\partial}_{\bar{j}}K$ are in different complex cohomology classes differ by F terms. For $\mathbb{C}P^{N-1}$, $\dim H^{1,1}(M, \mathbb{R}) = 1$ and the Kähler class is specified by a single parameter. We can take

$$K(\Phi, \Phi^\dagger) = \frac{1}{g^2}\log(1 + \sum_{i=1}^{N-1} \Phi_i^\dagger \Phi_i). \tag{18}$$

The supersymmetric ground states of an $N = 2$ σ-model are in one-to-one correspondance with the complex cohomology classes of the target space, and by spectral flow so are the chiral primaries. Using semiclassical techniques to compute the chiral ring, one finds it to be a deformation of the classical cohomology ring: instantons can contribute to correlation functions of the chiral primaries. In simple cases the possible contributions are determined by the chiral anomaly. For $\mathbb{C}P^{N-1}$, the classical cohomology ring is the powers of the Kähler form x allowed on a $2(N-1)$-dimensional manifold, up to x^{N-1}. The instanton changes the relation $x^N = 0$ to

$$x^N = e^{-2\pi/g^2}, \tag{19}$$

which defines the chiral ring.

Many $N=2$ supersymmetric theories in two dimensions admit a Landau-Ginsburg description[18,32] if their superspace Lagrangian is of the form

$$\mathcal{L} = \int d^4\theta \sum_i \phi_i \bar{\phi}_i + \int d^2\theta W(\phi_i) + h.c. \tag{20}$$

where ϕ_i, $\bar{\phi}_i$ are the chiral and antichiral (a,c) superfields and the superpotential W is an analytic function of the complex superfields which obeys non-renormalization theorems. The ground states of the theory are $dW(\phi)=0$. The chiral ring is the ring of polynomials generated by the ϕ_i modulo the relations $dW(\phi)/d\phi_i = D\bar{D}\phi_i \sim 0$.

In this light, and for many other purposes, a more useful definition of the $\mathbb{C}P^{N-1}\sigma$-model is provided by a gauged $N=2$ model, which constructs $\mathbb{C}P^{N-1}$as a quotient of $\mathbb{C}^N$:

$$\mathcal{L} = \int d^4\theta \left[\sum_{i=1}^{N} \bar{S}_i e^{-V} S_i + \frac{N}{g^2} V \right]. \tag{21}$$

S_i are N chiral superfields which become the homogeneous coordinates on $\mathbb{C}P^{N-1}$. We have introduced a factor of N with the coupling $1/g^2$ which will make the $N \to \infty$ limit well defined.[6] V is a real vector superfield, whose components become the many auxiliary fields of the following component form of the Lagrangian:

$$\begin{aligned} \mathcal{L} &= \frac{N}{g^2}\{ \, (D_\mu n_i^*)^\dagger (D_\mu n_i) + \bar{\psi}^i (i\not{D} + \sigma + i\pi\gamma^5)\psi_i - \\ &\quad +(\sigma^2+\pi^2) - \lambda(n_i^* n_i - 1) + \bar{\chi} n_i^* \psi^i + \bar{\psi}^i n_i \chi \, \}. \end{aligned} \tag{22}$$

The superfields S have complex components n_i and ψ_i. (We rescaled them by $\sqrt{N}/g$.) The constraint $n_i^* n_i = 1$ is imposed by the Lagrange multiplier λ; the phase of n and ψ is gauged by A_μ (which appears in $D_\mu = \partial_\mu + iA_\mu$). The fields σ and π implement 4-fermi interactions and by the equations of motion are equal to $\bar{\psi}^i\psi_i$ and $i\bar{\psi}^i\gamma^5\psi_i$ respectively.

Integrating out the superfields S_i in (21), one obtains an effective action for the $\mathbb{C}P^{N-1}$models which has the form of a Landau-Ginsburg model. By gauge invariance only the field-strength superfields X and $\bar{X}$ remain since V is not gauge invariant:

$$S_{\text{eff}} = \frac{N}{2\pi}\int d^2x \left\{ \int d^2\theta W(X) + \int d^2\theta \bar{W}(\bar{X}) + \int d^4\theta [Z(X,\bar{X},\Delta,\bar{\Delta})] \right\} \tag{23}$$

with

$$W(X) = X(\log X^N - N + A(\mu) - i\theta). \tag{24}$$

A is a renormalized coupling and θ the instanton angle, and

$$X = D_L \bar{D}_R V, \qquad \bar{X} = D_R \bar{D}_L V. \tag{25}$$

The chiral ring is the powers of X mod $dW=0$: $\quad X^N = e^{-A+i\theta}$.

3.1. Topological-antitopological fusion equations and the new index

The action can be written in the following form

$$S = -\frac{N}{4\pi} \ln t \int d^2y d^2\theta \; x + \text{c.c.} \tag{26}$$

where $x = x(\Phi_i, \bar{\Phi}_i) = \bar{D}D \ln(1 + \sum \Phi_i \bar{\Phi}_i)$ represents the Kähler class, $\ln t \; x$ is the Kähler form and $\Phi_i, \bar{\Phi}_i$ are chiral superfields.

The chiral ring is generated by a single element X, the Kähler class, with relation

$$X^N = t^N \tag{27}$$

To write down the tt^* equations, we need to find the operator corresponding to a perturbation of t, and its action on the chiral ring $(X^{N-1}, \ldots, X, 1)$. It is represented by the matrix

$$C_t = \frac{N}{4\pi t} \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ t^N & 0 & 0 & \ldots & 0 & 0 \end{pmatrix}$$

The $\mathbb{Z}_N$ symmetry implies that the metric $g_{i\bar{j}} = \langle \bar{j} | i \rangle$ is diagonal. The metric g is a function only of $|t|^2$, because it is a path integral with total chiral charge zero, and chiral charge non-conservation is proportional to instanton number. Thus the equations become o.d.e.'s in terms of $|t|$. We can write them in terms of the dimensionless parameter $x = \beta t / 2\pi$, but to do this we need to take out the dimensional factors in $g_{i\bar{j}}$. Thus we define

$$q_j = \ln g_{j\bar{j}} + 2j \log |t| \qquad\qquad q_{j+N} \equiv q_j. \tag{28}$$

We will now use x as our coupling, and call it β in the following. The tt^* equation becomes

$$\frac{4}{N^2} \partial_\beta \partial_{\beta^*} q_i + e^{(q_{i+1} - q_i)} - e^{(q_i - q_{i-1})} = 0. \tag{29}$$

This equation is the affine $\hat{A}_{N-1}$ Toda equation.

A solution should be determined by the boundary conditions near $\beta \sim 0$ and $\beta \sim \infty$. In [1] these are found explicitly for the small β limit by a semiclassical calculation of the metric. For the large β limit the solution is exponentially small in β, the precise form of the leading exponential being determined by the soliton spectrum, which is already known for these (integrable) models. For the cases of $\mathbb{C}P^1$ and $\mathbb{C}P^2$, the tt^* equations become special cases of the Painlevé Ⅲ equation, and the connection formula between small and large β asymptotics is known.[3].

A reasonable ansatz for the large N limit would be that the metric and index are continuous functions of the variable $s \equiv i/N$. The large N effective action of

the $\mathbb{C}P^{N-1}$ model still has N fields with an action of the form $\exp(NS_{eff})$ at an appropriate saddle point. We therefore rescale $q_i \to \frac{1}{N}\, q_i$. The tt^* equation becomes

$$4\partial_\beta\partial_{\beta^*}H + \frac{\partial^2}{\partial s^2}e^H = 0, \tag{30}$$

with $H = q'$.

This equation has been studied in several contexts. It was first noted for a connection with 4D self-dual gravity.[19,20,21] More recently, it has been studied in the context of the large n limit of W_n algebra.[22,23,24] It is also a well known scaling limit of the two-dimensional infinite Toda lattice.[26] A formal solution of the boundary value (Goursát) problem for the equation has been given in [25].

We still need to specify the boundary conditions to select a solution to the equation. We can deduce the large N limit of our metric for small $|\beta|$ from the semi-classical result of Cecotti and Vafa[4] (this is essentially the two-point function $\langle\phi_i\bar{\phi}_j\rangle$ reduced to constant field configurations, or $\int d\varphi\ x^i \wedge *x^j$)

$$e^H = \frac{s(1-s)}{|\beta|^2(-\ln(|\beta|/2)-\gamma)^2} \tag{31}$$

(for $0 < s < 1$ and defined elsewhere by periodicity), where γ is Euler's constant, a factor predicted by the connection formula for the $n = 1, 2$ equations, and described in [4] as a one-loop correction to the semiclassical calculation.

For large β, each sector with one soliton of mass m satisfying the Bogomolnyi bound contributes $((f+1)-f)\exp -\beta m$ times a factor depending on its central charge Δ to the new index, and by (15) to H. We have N such solitons with mass $m = m_0$; the central charge is $\pi(+L) - \pi(-L)$ or one can just linearize (30) to see the appropriate boundary condition

$$H(s) \sim -\frac{\exp\left(-2\pi|\beta|\right)}{\sqrt{2\pi|\beta|}}\cos(2\pi s) \tag{32}$$

which satisfies our equation to first order. This limit is not a solution and one could use (30) to generate corrections to H coming from multi-soliton sectors.

The new index is

$$Q_{ab} = \frac{i\beta}{L}\mathrm{Tr}\ _{ab}\ (-1)^F\ F\ e^{-\beta H} \tag{33}$$

where a and b characterize the vacua at spatial infinity. We can rewrite it as a path integral with an insertion of the fermion number charge $\int dx^1 J^0_F$.

Recall now the action (21) and its component form (22). Since the fields S_i appear quadratically in (21), it is possible to integrate them out exactly, at least in terms of a one-loop determinant:

$$\begin{aligned}\mathcal{L} \;\; = \;\; & -N\mathrm{Tr}\ \log(-D_\mu D^\mu + \lambda) + N\mathrm{Tr}\ \log(i\not{D} + \sigma + i\pi\gamma^5) \\ & + \frac{N}{g^2}(\sigma^2 + \pi^2 - \lambda) + \text{fermionic}.\end{aligned} \tag{34}$$

In the large N limit, the N in front of the action means that the integration over auxiliary fields can be done by saddle point, and calculations at leading order in $1/N$ can be done by classical techniques.

The effective action (34) has been extensively studied at $T = 0$ [27] and at finite temperature (typically not in the supersymmetric context, but the results can be easily adapted)[29,30]. In the large N limit it is $O(N)$ and we can calculate bulk quantities like the free energy simply by extremizing it with respect to the auxiliary fields. The 'new index' is computed similarly, with the main differences being that we take periodic fermi boundary conditions (and have unbroken supersymmetry), we insert the fermion number operator F (this will be done by differentiating with respect to a coupling at the end), and we fix the boundary conditions at $x^1 = \pm L$ to go to two possibly different values of $\sigma + i\pi$ (the ground states being characterized by the phase of $\sigma + i\pi$).[12] This last condition means that we need to consider non-constant background fields in the functional integral. For general background fields this is quite complicated, but what saves us is that the required variation is small, of $O(1/N)$, so we only need the leading terms in an expansion in derivatives and amplitude. The derivative terms (to the accuracy we need them) are

$$S_{eff} = N\int d^2x \frac{1}{8\pi m_0^2}\left(F_{\mu\nu}^2 + (\partial_\mu\sigma)^2 + (\partial_\mu\pi)^2\right) + \frac{i}{2\pi}\epsilon^{\mu\nu}F_{\mu\nu}\mathrm{Im}\log(\sigma+i\pi) + V_{eff} + \quad (35)$$

The finite temperature effective potential is

$$\begin{aligned} V_{eff} &= \frac{1}{g^2}(\sigma^2+\pi^2) - \sum_{k^0}\int \frac{dk^1}{(2\pi)^2}\mathrm{tr}\ln[k_\mu\gamma^\mu - A_\mu\gamma^\mu - (\sigma + i\pi\gamma^5)] \\ &\quad -\frac{1}{g^2}\lambda + \sum_{k^0}\int \frac{dk^1}{(2\pi)^2}\ln[(k_\mu - A_\mu)^2 + \lambda] \qquad (36) \\ &= V_{eff}|_{T=0} - \int_0^\infty \frac{dk}{2\pi}\ln|1 - e^{-\beta\sqrt{k^2+\lambda}}e^{i\beta A_0}|^2 \\ &\quad + \int_0^\infty \frac{dk}{2\pi}\ln|1 - e^{-\beta\sqrt{k^2+\sigma^2}}e^{i\beta A_0}|^2. \qquad (37) \end{aligned}$$

This is essentially the standard expression from statistical mechanics of a free field [31] (with chemical potential iA_0) with one difference: we incorporated the periodic fermion boundary conditions, which led to the sign change in (37).

We will not go into the details to obtain the new index, but just state here the results.[12] The index is obtained by the saddle point approximation

$$Q(s,\beta) = -\frac{N}{2\pi i}\frac{d}{ds}\frac{\beta}{NL}S_F(\beta,s)|_{min} \qquad (38)$$

with

$$\begin{aligned} \frac{\beta}{NL}S_F &= -\frac{1}{2\pi}(2\pi(s-\frac{1}{2}) + \beta A_0)^2 + \beta^2\sigma^2\frac{1}{4\pi}\left(\ln\frac{\sigma^2}{m_0^2} - 1\right) - \qquad (39) \\ &\quad \beta\int_0^\infty \frac{dk}{2\pi}\ \ln\left|1 - e^{-\beta\sqrt{k^2+\sigma^2}+i\beta A_0}\right|^2. \end{aligned}$$

We still need to minimize the effective action with respect to the auxiliary fields σ and A_0. It turns out that the minimization can be reinterpreted as precisely a combination of known techniques for finding solutions to self-dual gravity from those for simpler equations via Legendre transform. In this way we can prove that the index and metric, computed independently, solve the tt^* equation. We refer the interested reader to [12].

4. The Grassmannian models

Now we discuss non-linear σ-models on complex Grassmannian target manifolds $\mathbb{G}(k,N)$. These spaces have (complex) dimension $k(N-k)$ and consist of all k dimensional subspaces of the complex vector space $\mathbf{C}^N$.

The classical cohomology[33,35,36,34] of the Grassmannian manifold $\mathbb{G}(k,N)$ is generated from the Chern classes X_i, where X_i is a (i,i) form, with the relations

$$\sum_{i\geq 0} X_i t^i \ . \ \sum_{j\geq 0} Y_j t^j = 1 \tag{40}$$

where $Y_j's$ are some functions.

The quantum cohomology ring results from a modification to relations (40) of the form[34]

$$(\sum_{i=0}^{k} X_i \ t^i) \ . \ (\sum_{j=0}^{N-k} Y_j \ t^j) = 1 + (-1)^{N-k} \ t^N \tag{41}$$

which imply $Y_{N+1-i} + (-1)^{N-k} \ \delta_{i,1} = 0, \quad 1 \leq i \leq k$.

The quantum cohomology ring is generated by polynomials in the X_i's (if one eliminates the Y_j's) subject to the constraints (41), and has dimension $N!/k!(N-k)!$.

As quantum field theories, the Grassmannian σ-models (for $k \geq 2$) can be thought of as generalizations of the $\mathbb{C}P^{N-1}$models. The Lagrangian has a similar form

$$\mathcal{L} = \int d^4\theta \left[\sum_a \bar{S}_a e^{-V} S_a + \alpha \mathrm{Tr}\ V \right] \tag{42}$$

where now the chiral fields S_{ia} carry two indices, a 'gauge' $U(k)$ index i, and a 'flavour' $SU(N)$ index a, since there are $(N\times k)$-matrix scalar fields $n = (n_i^a)$ and $(N\times k)$-matrix Dirac spinor fields $\psi = (\psi_i^a)$. V is a $k \times k$ matrix of superfields with gauge group $U(k)$ (see [38]).

Integrating out the superfields, one recovers again a Landau-Ginsburg-type action:[3] The field-strength superfields (which we now call λ, $\bar{\lambda}$) belong to the adjoint representation of $U(k)$ and are now gauge covariant. The real gauge-invariant objects of (42) are the Ad-invariant polynomials in the field-strengths λ, with a ring generated by the (a,c) superfields $X_i \quad (i = 1,2,\ldots,k)$ defined by

$$\det[t-\lambda] = t^k + \sum_{j=1}^{k} (-1)^j t^{k-j} X_j. \tag{43}$$

At the topological field theory level one can assume that the λ's and $\bar{\lambda}$'s are independent of each other and therefore, as matrices, are diagonalizable and all their eigenvalues λ_m are distinct. Then, without loss of generality, the functional determinants in the path integral are the same as for the $\mathbb{C}P^{N-1}$case, since the full background is now abelian, and the superpotential, which fixes the theory completely, becomes

$$W_f(\lambda_1, \lambda_2, \ldots, \lambda_k) = \frac{1}{2\pi} \sum_{j=1}^{k} \lambda_j (\log \lambda_j^N - N + A(\mu) - i\theta). \tag{44}$$

The gauge-invariant fields are now polynomials in the eigenvalues λ_m of the field-strengths λ and are generated by the elementary symmetric functions

$$X_i(\lambda) \equiv \sum_{1 \leq l_1 < l_2 < \ldots l_i \leq k} \lambda_{l_1} \lambda_{l_2} \ldots \lambda_{l_i} \quad (i = 1, \ldots, k) \tag{45}$$

The ring relations are $\lambda_j^N = \text{const.}$ and the quantum cohomology of the Grassmannian σ-models will be generated by the elementary symmetric functions X_i's. As quantum field theories, the Grassmannian σ-models $\mathbb{G}(k, N)$ can thus be identified as the tensor product of k copies of the $\mathbb{C}P^{N-1}\sigma$-model with certain redundant states eliminated.[3]

We are interested in finding a suitable basis for the metric of the Grassmannian σ-model. In view of (44,45), a basis for the chiral ring will consist of polynomials $P_r(X_i)$ (for $r = 1, \ldots, N!/k!(N-k)!$) in the X_i's. Then one can determine C in (15) from the ring relations and derive the tt^* equations. However the equations are difficult to handle in this form. A more enlightening way to obtain and study the equations is to use the map (45). The metric is then defined in terms of the variables λ_m, in the following way

$$\langle P_r(X_i) | P_s(X_j) \rangle = \frac{1}{k!} \langle \Delta(\lambda) P_r(X_i(\lambda)) | \Delta(\lambda) P_s(X_j(\lambda)) \rangle_f \tag{46}$$

where $\Delta(\lambda)$ is the Vandermonde determinant and is the Jacobian $J = \det(\partial X_i / \partial \lambda_j)$ of the transformation in (45).

Each basis element $P_s(X_j)$ can then be written as a polynomial $\Delta(\lambda) P_s(X_j(\lambda))$ in the different λ_m's. The metric on the LHS of (46) is a sum over the products of the metrics for the $\mathbb{C}P^{N-1}$models, which are diagonal $\langle \overline{\lambda_m^k} | \lambda_m^s \rangle = \delta_{ks} \langle \overline{\lambda_m^k} | \lambda_m^s \rangle$.

The ground-state metric is in general non-diagonal and complicated for an arbitrary basis. However, the form of the metric suggests that one might try finding an orthogonal basis in which the metric becomes diagonal and simple (i.e. with each component consisting of a single term). This basis is the 'flat coordinates' basis, in which the tt^* equations have the simplest form[1]. This basis is characterized by a two-point function η which is independent of the perturbing parameters of the model and squares to 1 ($\eta^* = \eta^{-1} = \eta$).

For example, for $\mathbb{G}(2, N)$, such a basis is given by the $N(N-1)/2$ elements $|mn\rangle \equiv |\lambda^m \lambda^n\rangle$ with $n > m$, such that $|mn\rangle = -|nm\rangle$ and $|mn\rangle = 0$ if $m = n$.

The metric is defined as $\langle mn|mn\rangle = \langle m|m\rangle\langle n|n\rangle = -\langle mn|nm\rangle = \langle nm|nm\rangle$, where $\langle m|m\rangle = \langle\lambda^m|\lambda^m\rangle$ is the (diagonal) metric element for $\mathbb{C}P^{N-1}$.

It follows that the tt^* equations for $\mathbb{G}(k,N)$, for any k, can be derived from the ones for the $\mathbb{C}P^{N-1}$ model. Defining

$$\begin{aligned} q_{l_1 l_2 \ldots l_k} &= \ln g_{\overline{l_1 l_2 \ldots l_k}, l_1 l_2 \ldots l_k} - \frac{2\sum_i l_i - k(N-1)}{2N} \log|\beta|^2 \\ &= q_{l_1} + q_{l_2} + \ldots + q_{l_k}. \end{aligned} \tag{47}$$

The tt^* equations become

$$\partial_{\bar z}\partial_z q_{l_1 l_2 \ldots l_k} + \sum_{i=1}^{k} \Big\{ \exp[q_{l_1 l_2 \ldots l_{i+1} \ldots l_k} - q_{l_1 l_2 \ldots l_k}] - \exp[q_{l_1 l_2 \ldots l_k} - q_{l_1 l_2 \ldots l_{i-1} \ldots l_k}] \Big\} = 0 \tag{48}$$

with

$$0 \le l_1 < l_2 < \ldots < l_i < \ldots < l_k \le N-1, \tag{49}$$

and any exponential containing any q with 2 indices the same is ignored.

We now discuss the boundary conditions, which are derived from the UV and IR limits of those for the $\mathbb{C}P^{N-1}$model. For $\mathbb{C}P^{N-1}$, all the fundamental one-soliton contributions to the tt^* metric in the IR can be determined

$$q_i = \sum_{r=1}^{N-1} \binom{N}{r} \sin\Big[\frac{2\pi r}{N}(i+\frac{1}{2})\Big]\frac{1}{\pi} K_0(m_r\beta) + \ldots \tag{50}$$

for $i = 0, \ldots, N-1$, with masses $m_r = 4N|\beta|^{\frac{1}{N}} \sin\left(\frac{\pi r}{N}\right)$.
Then the boundary conditions in the IR for $\mathbb{G}(k,N)$ are simply given by

$$q_{l_1 l_2 \ldots l_k} = q_{l_1} + q_{l_2} + \ldots q_{l_k} \tag{51}$$

This allows to obtain the soliton spectrum and soliton numbers between various vacua from the canonical basis,[3,13] the basis of vacua for $\mathbb{G}(k,N)$. For example we find that, for $\mathbb{G}(2,4)$, solitons fall into multiplets of completely antisymmetric representations of $SU(4)$, the representations and (the two different) masses being determined by how far apart the respective six vacua are (see [13]).

We are now interested in checking whether the tt^* equations become solvable in the large N and k limit, in the same way as they were for the large N $\mathbb{C}P^{N-1}$model.[12] Consider first the equations for $\mathbb{G}(2,N)$ in the large-N limit. There is an immediate generalization to $\mathbb{G}(k,N)$ for any k. Again we can assume that the metric becomes a continuous function of two variables $s_1 \equiv i/N$, and $s_2 \equiv j/N$. Redefining

$$q_{ij} = \frac{1}{N}\ln\langle\bar i|i\rangle\langle\bar j|j\rangle + 2\frac{(i+j)}{N}\log|\beta|^2, \tag{52}$$

the tt^* equations become

$$\frac{4}{N}\partial_{\bar\beta}\partial_\beta\ q_{ij} + e^{N(q_{ij+1}-q_{ij})} - e^{N(q_{ij}-q_{ij-1})} + e^{N(q_{i+1j}-q_{ij})} - e^{N(q_{ij}-q_{i-1j})} = 0 \tag{53}$$

with $q_{i+N,j} = q_{ij}$ and $q_{i,j+N} = q_{ij}$. Then, expanding, the equation reduces to

$$4\partial_{\bar{\beta}}\partial_{\beta}\, q(s_1, s_2) = \frac{\partial}{\partial s_1} \exp\Big[\frac{\partial q}{\partial s_1}\Big] + \frac{\partial}{\partial s_2} \exp\Big[\frac{\partial q}{\partial s_2}\Big] \tag{54}$$

with general solution $q(s_1, s_2) = \ln g(s_1, s_2) = \ln[g(s_1)g(s_2)] = q(s_1) + q(s_2)$, where $s_2 > s_1$, and $q(s_i)$ is the solution for the $\mathbb{C}P^{N-1}$model in the large N limit.

The generalization for arbitrary (finite or infinite) k is, for $\mathbb{G}(k, N)$

$$\partial_{\bar{z}}\partial_z\, q(s_1, s_2, \ldots, s_k) = \sum_{i=1}^{k} \frac{\partial}{\partial s_i} \exp\Big[\frac{\partial q}{\partial s_i}\Big], \tag{55}$$

with solution $q(s_1, s_2, \ldots, s_k) = \ln g(s_1, s_2, \ldots, s_k) = \ln[g(s_1)g(s_2)\ldots g(s_k)]$, and $s_1 < s_2 < s_3 < \ldots < s_k$.
We note that there is no particular difficulty in going beyond $k = 1$.

The Grassmannian σ-models have not been solved completely as quantum field theories and these results should provide further insights into them.

5. The perturbed A_n Minimal models

The flow of unitary models towards other massless unitary models were first studied in perturbation theory.[14,15] It was shown that perturbing the $(m+1)^{\text{th}}$ minimal model (for large m) by the least relevant operator would make it flow in the IR to the m^{th} minimal model, as the coupling to this operator grows. In this way, the authors found a flow between successive minimal models. The extension of these results to the RG flow in $N = 1$ discrete series was discussed in [16]. These integrable deformations have also been studied using the TBA[17].

Here we look at the tt^* formalism for the $N = 2$ minimal models $W = X^{n+1}/n+1$ perturbed by different supersymmetry preserving operators.[1] By considering various perturbations tX^{k+1}, it is possible to study non-perturbatively (for large n) the flow between minimal models and other minimal models and massive models. The integrable models for the minimal models perturbed by the two most-relevant supersymmetry preserving operators can be solved completely, and the application to non-integrable perturbations is briefly discussed.

Consider first the Landau-Ginsburg potential corresponding to the $N = 2$ A_n series deformed by the most relevant chiral operator[1]

$$W(X, t) = \frac{X^{n+1}}{n+1} - tX \tag{56}$$

where X is the non-trivial chiral field of lowest dimension.

The ring is identical to the one for the $\mathbb{C}P^{N-1}$model, and the tt^* equations are thus of the same affine Toda form

$$\partial_z\partial_{z^*} q_i + e^{(q_{i+1} - q_i)} - e^{(q_i - q_{i-1})} = 0 \tag{57}$$

with

$$
\begin{aligned}
q_i &= \log\langle\bar{i}|i\rangle - \frac{2i-n+1}{2n}\log|t|^2 \\
z &= \frac{n}{n+1}t^{\frac{(n+1)}{n}} \\
q_i &\equiv q_{i+n}, \quad 0 = q_i + q_{n-i-1}.
\end{aligned}
\tag{58}
$$

We will again consider the large n limit of these equations. We note now the difference with $\mathbb{C}P^{N-1}$. For the $\mathbb{C}P^{N-1}$ model, the large N effective action still has N fields in the theory and the q_i were rescaled by $1/N$. Here we do not need this rescaling. To get a smooth large n limit we redefine the q_i's in terms of a continuous variable $s \equiv \frac{i}{n}$,

$$
\begin{aligned}
q_i &\rightarrow q(s,|t|) = \log g(s,|t|) - (2s-1)\log|t|, \\
z &\rightarrow nt
\end{aligned}
\tag{59}
$$

The tt^* equations become $\quad \partial_t\partial_{t^*}\frac{1}{n}F + \frac{\partial^2}{\partial s^2}e^{\frac{1}{n}F} = 0,$
with $F = \frac{\partial q}{\partial s}$. The exponential is of order $\frac{1}{n}$, and we get

$$
\partial_t\partial_{t^*}F + \frac{\partial^2}{\partial s^2}F = 0, \tag{60}
$$

the ordinary 3D Laplace equation. The general solution for the metric is

$$
q(s,t) = -\sum_{p\geq 1}\frac{c_p \sin 2\pi ps}{\pi p}\ K_0(4\pi p|t|) + (as+b)\log|t| \tag{61}
$$

where c_p is the soliton multiplicity and is 1 (see [44]).
In the limit where $t \rightarrow 0$

$$
q(t,s) \rightarrow -\sum_{p\geq 1}\frac{\sin 2\pi ps}{\pi p}\ [-\ln(2\pi p|t|) - \gamma] + (as+b)\log|t| \tag{62}
$$

Then, we can use the expression[11]

$$
\sum_{p\geq 1}\frac{\ln p}{p}\ \sin 2\pi ps = \pi\ln\Gamma(s) + \frac{\pi}{2}\ln(\sin\pi s) - \frac{1}{2}\pi(1-2s)(\gamma+\ln 2) - \pi(1-s)\ln\pi \tag{63}
$$

for $0 < s < 1$, and

$$
\sum_{p\geq 1}\frac{\sin 2\pi ps}{p} = \pi(k+\frac{1}{2}-s) \qquad \text{for} \quad 2k\pi < 2\pi s < (2k+1)\pi. \tag{64}
$$

We have $\quad q(t,s) = \ln\Gamma^2(s)\sin\pi s - \ln\pi - (2s-1)\log|t| + (as+b)\log|t|$

$$
\text{and} \qquad g(s,t) = \frac{1}{\pi}\Gamma^2(s)\sin\pi s \tag{65}
$$

for $a=b=0$. This is consistent with the known result[1] for the ground state metric at $t=0$

$$g_{\bar{i}i}(t\to 0)=\frac{\Gamma(\frac{i+1}{n+1})}{\Gamma(1-\frac{i+1}{n+1})}, \tag{66}$$

which for large n becomes (65). In the IR (large t), we have

$$q\sim -\sum_p \sin 2\pi ps\frac{e^{-4\pi p|t|}}{\sqrt{8\pi^2p^3|t|}}+.. \tag{67}$$

We can now compare this with the results in [1,13] obtained from semi-classical considerations

$$q_i\sim -\sin\left[\frac{2\pi}{n}(i+\frac{1}{2})\right]\frac{e^{-4n|t|\sin\frac{\pi}{n}}}{\sqrt{8\pi n|t|\sin\frac{\pi}{n}}}. \tag{68}$$

In the large n limit this is $\quad -\sin(2\pi s)\frac{e^{-4\pi|t|}}{\sqrt{8\pi^2|t|}}, \quad$ the first term in (67).

In order to derive the 'new index', we write the metric in terms of a dimensionless quantity, $z=M\beta$, where $M=4\lambda$ is the mass of the fundamental soliton[44] and β is the perimeter of the cylinder (set to 1 up to now). This is done by rescaling the superfields $X\to t^{\frac{1}{n}}X$; then $W\to\lambda W$ with $\lambda=t^{\frac{1+n}{n}}$ and $q_i(z)=\log\langle\bar{i}|i\rangle$ with $\det[g]=1$. The action of the renormalization group can be seen as this rescaling of the superpotential by λ. Then $\lambda\to 0$ corresponds to the UV limit and $\lambda\to\infty$ to the IR.

In the large n limit, $\lambda\to t \quad$ and $\quad q(s,z)=\log g(s,z), \quad$ and the ground-state metric obeys

$$16\partial_z\partial_{z^*}\frac{\partial q}{\partial s}+\frac{\partial^2}{\partial s^2}\frac{\partial q}{\partial s}=0, \tag{69}$$

and $$q(s,z)=-2\sum_{p\geq 1}\frac{\sin 2\pi ps}{\pi p}\ K_0(\pi p|z|)+... \tag{70}$$

The dependence of the metric on the coupling, as $z\to 0$, $g\to\frac{1}{\pi}\Gamma^2(s)\sin\pi s\ |z|^{1-2s}$, should be consistent with the result[1] that in the basis of ground-state vacua of definite charge the metric behaves as $\quad g_{\bar{i}i}\sim|z|^{-2q_i}, \quad$ where q_i is the charge of the i-th Ramond vacuum. Thus we find the Ramond charge to be, in the large n limit, $q^R_s=s-\frac{1}{2}$. The Neuveu-Schwarz $U(1)$ charges for the $N=2$ minimal models are $q_k=\frac{k}{(n+1)}$ for $k=1,...,n$, which become $q_s=s$, and since the Ramond charges are $q^R_k=q_k-\hat{c}/2$, this tells us that $\hat{c}=c/3$ or $c=3$.

We now look at the index Q. In the conformal limit, the eigenvalues Q_{ab} of the index should be the same as the $U(1)$ charges for the superfields $|X^k\rangle$ in the Ramond sector for the $N=2$ minimal models.
The index $Q(s,z)$ is simply[2]

$$Q(s,z)=-\frac{z}{2}g^{-1}\partial_z g=-\sum_{p\geq 1}z\sin 2\pi ps\ K_1(\pi p|z|) \tag{71}$$

We see that in the IR, we get a series of terms in $e^{-pM\beta}$ each of which is the contribution of a single massive particle saturating the Bogolmonyi bound $m = 2p|\Delta W|$. There are no corrections from multi-solitons states, which seem to suggest that the particles remain free. Actually this is not too surprising since the coupling and non-trivial scattering for large n is of order $O(1/n)$.

In the UV limit $z \to 0$,

$$Q(s, 0) \to -\sum \frac{\sin 2\pi ps}{\pi p} \to s - \frac{1}{2} \tag{72}$$

for all s, which is the desired result. This is also consistent with the well-known result that the ground-state energy for the minimal models is given by[28]

$$E(\beta) = -\frac{\pi}{6\beta}\left[\frac{3(n-1)}{(n+1)}\right] = \frac{2\pi}{\beta}(-\frac{c}{12}) \tag{73}$$

which for large n implies $c = 3$.

As mentionned earlier, at first order the particles seem free. However we can generate corrections of order $\frac{1}{n}$. Call $F^{(0)}$ the first order solution of (60)

$$16\partial_z\partial_{z^*}F^{(0)} + \frac{\partial^2}{\partial s^2}F^{(0)} = 0 \tag{74}$$

with

$$F^{(0)} = -4\sum \cos 2\pi ps \; K_0(\pi p|z|) \tag{75}$$

The next order correction in the exponential expansion will be a solution of

$$\frac{4}{z}\partial_z z\partial_z F^{(1)} + \frac{1}{2}\frac{\partial^2}{\partial s^2}[F^{(0)}]^2 = 0 \tag{76}$$

Integrating, using formulas found in [45],

$$F^{(1)} = \sum_{p,p'} \frac{8\sin 2\pi ps \;\; \cos 2\pi p's}{(p^2 - p'^2)}\left[pp'K_0(\pi p|z|)K_1(\pi p'|z|) + p^2 K_0(\pi p'|z|)K_1(\pi p|z|)\right] \tag{77}$$

Integrating over s, one finds

$$q^{(1)} = \sum_{p,p'} \frac{-4}{(p^2 - p'^2)}\left[\frac{\cos 2\pi s(p+p')}{(p+p')} + \frac{\cos 2\pi s(p-p')}{(p-p')}\right]. \tag{78}$$

$$.\left[pp'K_0(\pi p|z|)K_1(\pi p'|z|) + p^2 K_0(\pi p'|z|)K_1(\pi p|z|)\right] \tag{79}$$

which gives the two-particle interactions.[2] In such a way, it is possible to generate successive corrections to the index of order $\frac{1}{n}$ and determine the S-matrix.

So far we have looked at the $N = 2$ Minimal models perturbed by the most relevant chiral field. Consider now a general perturbation of the form

$$W = \frac{X^{n+1}}{n+1} - t\frac{X^{k+1}}{k+1} \tag{80}$$

The critical points are $X^n = tX^k$ and $X = 0$ is degenerate of order k. As we move away from the conformal theory at $t = 0$, the critical points spread out, and in the IR limit, for large t, we have a set of isolated critical points. The flow generated by the original model has taken us to another massless model in the IR limit around $X = 0$ (with $W = X^{k+1}$). Around the other critical points the models are massive in the IR limit and defined by $W = \frac{X^{n-k+1}}{n-k+1} - tX$.

The tt^* equations can be studied for any direction of perturbation, however only very special directions lead to integrable quantum field theories and these cases produce tt^* equations of the affine Toda type.[1] These equations always lead to massive theories in the IR. For $k > 1$, the models are not integrable and the tt^* equations become more complicated, also, solitons between vacua do not always saturate the Bogolmonyi bound.

For a general perturbation X^{k+1}, the coupling matrix C_t which characterizes the chiral ring by multiplication by the operator $-\partial W/\partial t$ has components

$$\begin{aligned} C_{ij} &= -\frac{1}{k+1}\delta_{j-i,k+1} \qquad \text{for} \quad i = 1, 2, \ldots, n-1-k \qquad (81)\\ &= -\frac{t}{k+1}\ \delta_{j-i,2k+1-n} \qquad \text{for} \quad i = n-k, \ldots, n. \qquad (82)\end{aligned}$$

The model is invariant under the symmetry $\quad X \to e^{\frac{2\pi i}{n-k}}X, \quad$ which leaves the chiral elements $\{1, X^{n-k}\}$ invariant. Under this symmetry, the elements $\{X^l, X^{n-k+l}\}$ also transform in the same way. Thus the metric $g_{\bar{h},k} = \langle h|\bar{k}\rangle$ is diagonal except for elements $\langle l|n-k+l\rangle$.

The topological metric is the 2-point correlation function

$$\eta_{ij} = \langle \phi_i\phi_j\rangle_{top} = \langle i|j\rangle = \mathrm{Res_W}[\phi_i\phi_j] \tag{83}$$

with

$$\mathrm{Res_W}[\phi] = \frac{1}{(2\pi i)^n}\int_\Gamma \frac{\phi(X)dX^1\wedge\ldots\wedge dX^n}{\partial_1 W\partial_2 W\ldots\partial_n W} \tag{84}$$

And we have

$$\eta_{ij} = \delta_{i+j,n-1} + t\ \delta_{i+j,2n-1-k} + t^2\ \delta_{i+j,3n-1-2k} + \ldots \tag{85}$$

The first perturbation, $tX^2/2$, is integrable[46] and the finite n case has been discussed in [1]. The topological metric $\eta_{ij} = \delta_{i+j,n-1} + t\ \delta_{i+j,2n-2}$. In the large n limit, the model can be solved completely, the tt^* equations reducing again to the Laplace equations (with different coefficients). We will not discuss the solution in detail

but just mention that one needs to define two sets of equations since the equations couple metrics of odd i together and metrics of even i together.

Going beyond $k=1$ becomes more complicated. It is not yet clear whether the equations admit a nice solution in the large n limit. Consider the perturbation $W=\frac{X^{n+1}}{n+1}-t\frac{X^3}{3}$. The boundary conditions are known. In the UV, we have (65). In the IR, we should expect a combination of (67) for the massive model and (66) (with $i=0,1$) for the massless model.
We can choose to work in the 'flat coordinates' basis[48,47]

$$\{1, X, X^2, \ldots, X^{n-2}-\frac{t}{3}, X^{n-1}-\frac{2}{3}Xt\} \tag{86}$$

The elements $\{1, X^{n-2}\}$ and $\{X, X^{n-1}\}$ change in the same way under $X \to \exp(\frac{2\pi i}{n-2})X$ and the reality constraints (resulting from the relation $\eta^{-1}g(\eta^{-1}g)^*=1$ between g and η) imply

$$g_{0\overline{n-2}}g_{1\bar{1}}+g_{0\bar{0}}g_{1\overline{n-1}} = 0; \quad g_{\overline{n-2}0}g_{n-1\overline{n-1}}+g_{n-2\overline{n-2}}g_{n-2\bar{0}}=0 \tag{87}$$
$$g_{n-2\overline{n-2}}g_{1\bar{1}}+g_{n-2\bar{0}}g_{1\overline{n-1}} = 1; \quad g_{n-2\overline{n-2}}g_{1\bar{1}}+g_{n-2\bar{0}}g_{1\overline{n-1}}=1 \tag{88}$$

The tt^* equations then become more complicated. For example the first equation is

$$\partial_{\bar{t}}\left(g_{0\bar{0}}\partial_t g_{n-1\overline{n-1}}+g_{0\overline{n-2}}\partial_t g_{n-1\bar{1}}\right)=\frac{1}{9}\left[g_{3\bar{3}}g_{n-1\overline{n-1}}+\frac{2}{3}\bar{t}g_{3\bar{3}}g_{n-1\bar{1}}\right]. \tag{89}$$

It is not obvious at this point how or if the equations decouple in these cases. We hope to study this question further.

6. Conclusion

We have seen here that the tt^* formalism is a powerful tool in unravelling non-trivial dynamical information about a full quantum field theory from its topological data. The formalism is completely solvable for various models with a large number of superfields. One could hope that these methods will also have applications in $D>2$.

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